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# Some Remarks on Relative Chebyshev Centers

V. S. Balaganskii

*Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, Ekaterinburg, Russia*

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In this work we study a relative Chebyshev center of $K$ with respect to $Y$, where $K$ is a closed bounded convex subset of a Hilbert space $X$, and $Y$ is a closed convex subset of $X$. Some results of Amir and Mach [*J. Approx. Theory* **40**, (1984), 364–374] are extended. © 1997 Academic Press

We use the following notation and definitions: let $X$ be a normed linear space, $X^*$ the dual of $X$, and $\mathscr{F}(X)$ the set of all closed non-empty subsets of $X$; $B(x, r) = \{z \in X\colon \|z - x\| \leqslant r\}$, $S(x, r) = \{z \in X\colon \|z - x\| = r\}$, $S = S(0, 1)$, and $S^* = \{f \in X^*\colon \|f\| = 1\}$. Suppose that $Y, K \in \mathscr{F}(X)$, and $K$ is bounded. A nonnegative real number $R_Y(K)$ is called the relative Chebyshev radius of $K$ with respect to $Y$ if $R_Y(K)$ is the infimum of all numbers $r > 0$ for which there exists $y \in Y$ such that $K$ is contained in the ball $B(y, r)$. Any point $y \in Y$ for which $K \subset B(y, R_Y(K))$ is called a relative Chebyshev center of $K$ with respect to $Y$. We denote the set of all relative Chebyshev centers of $K$ with respect to $Y$ by $Z_Y(K)$; $Z(K) = Z_X(K)$, $R(K) = R_X(K)$, $R(y, K) = R_{\{y\}}(K)$; $P_Y(x) = \{z \in Y\colon \|x - z\| = \inf\{\|x - y\|\colon y \in Y\}\}$, and $P_Y$ is the metric projection onto $Y$.

In this work we prove some assertions concerning characterization of relative Chebyshev centers. In Section 1 the main result of the paper is established that in Hilbert space $X$ for a convex $Y \in \mathscr{F}(X)$ and a convex bounded $K \in \mathscr{F}(Y)$ the relation $Z_Y(K) \in P_Y(K)$ holds. This extends Corollary 2.9 of Amir and Mach [1], who assume that $K$ is compact and convex. In Section 2 we give necessary and sufficient conditions for $y$ to be the relative Chebyshev center of a convex bounded $K \in \mathscr{F}(X)$ with respect to a convex $Y \in \mathscr{F}(X)$, provided that $X$ is Hilbert. When $Y$ is a subspace, these results are equivalent to those of [1].



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## 1. MAIN RESULT

To prove Theorem 1 we need the following Lemmas 1 and 2 and a theorem of A. L. Garkavi.

LEMMA 1. *Let $X$ be a Banach space, $Y \in \mathscr{F}(X)$ a convex set, and $K \in \mathscr{F}(X)$ bounded. If $z \in Z_Y(K) \setminus Z(K)$, then there exists a functional $f \in S^*$ such that*

$$f(z) = \sup\{f(v): v \in Y\} = \inf\{f(v): v \in X,\ R(v, K) \leqslant R(z, K)\},$$

$$z \in Z_H(K), \qquad z \in Z_N(K),$$

*where*

$$H = \{v \in X: f(v) = f(z)\}, \qquad N = \{v \in X: f(v) \leqslant f(z)\}.$$

*Proof.* If $z \in Z_Y(K) \setminus Z(K)$, then $R(z, K) < \inf\{R(v, K): v \in X)\}$. By the triangle inequality the function $R(\cdot, K)$ is convex and continuous; hence, the set $D = \{x \in X: R(x, K) < R(z, K)\}$ is non-empty, convex, and open. Applying the Hahn–Banach theorem and the fact that $(z, x] \subset D$ for $x \in D$, we obtain a functional $f \in S^*$ such that

$$\sup\{f(v): v \in Y\} = \inf\{f(v): v \in D\} = f(z).$$

Since $z \in H \subset N$, we have $z \in Z_N(K)$ and $z \in Z_H(K)$.

THEOREM A. (A. L. Garkavi [2]). *Let $X$ be a Banach space*, $\dim X \geqslant 3$. *If every three-point set of $X$ has Chebyshev center belonging to its affine hull then $X$ is Hilbert.*

The following lemma is an easy consequence of Theorem A.

LEMMA 2. *Let $X$ be a Banach space*, $\dim X \geqslant 3$. *If $X$ is not Hilbert, then there are three points: $x_1, x_2, x_3$ such that for $Y = \operatorname{span}\{x_1, x_2, x_3\}$, $N = \operatorname{conv}\{x_1, x_2, x_3\}$, and $K = \operatorname{conv}\{0, x_1, x_2, x_3\}$ we have*

$$\dim Y = 3, \qquad 0 \in Z_Y(N) \cap Z_Y(K) \cap Z_K(K),$$

$$R(0, K) < \inf\{R(v, K): v \in \operatorname{conv}\{x_1, x_2, x_3\}\}.$$

*Proof.* Since $X$ is not Hilbert and $\dim X \geqslant 3$, there is a non-Hilbert subspace $Y \subset X$, $\dim Y = 3$. By Theorem A there exist three points $x_1, x_2, x_3 \in Y$ such that $Z(N) \cap A = \varnothing$, where $A = \operatorname{aff}\{x_1, x_2, x_3\} = \operatorname{aff} N$. Obviously $\dim A = 2$ and there exists a point $w \in Y$ such that $R(w, K) = \inf\{R(u, K): u \in Y\}$. Without loss of generality it may be assumed that $w = 0$. Then for $N$, $Y$, of and $K$ all the assertions of the lemma hold.

In [1, Corollary 2.9] it was proved that, in a Hilbert space $X$, if $Y \in \mathscr{F}(X)$ is a convex set and $K \in \mathscr{F}(X)$ is convex and compact, then $Z_Y(K) \in P_Y(K)$.

The following theorem improves this result.

THEOREM 1. *Let $X$ be a Banach space,* $\dim X \geqslant 3$. *The following statements are equivalent*:

(i) *$X$ is a Hilbert space.*

(ii) *For every convex set $Y \in \mathscr{F}(X)$ and for every convex bounded set $K \in \mathscr{F}(X)$ we have $Z_Y(K) \subset P_Y(K)$.*

(iii) *For every two-dimensianal subspace $Y \subset X$ and for every convex bounded set $K \in \mathscr{F}(X)$,* $\dim K \leqslant 2$, *we have $Z_Y(K) \subset \overline{P_Y(K)}$.*

*Proof.* (i) $\Rightarrow$ (ii). Suppose that the contrary holds: $z = Z_Y(K)$, but $z \notin P_Y(K)$. By Lemma 1 one can find $f \in S$ such that

$$\begin{aligned}\langle f, z\rangle &= \sup\{\langle f, v\rangle : v \in Y\}\\ &= \inf\{\langle f, v\rangle : v \in X,\ R(v, K) \leqslant R(z, K)\}.\end{aligned}$$

Let

$$\begin{aligned}H &= \{x \in X : \langle f, x\rangle = \langle f, z\rangle\},\\ N &= \{x \in X : \langle f, x\rangle \leqslant \langle f, z\rangle\},\\ l &= P_N^{-1}(z).\end{aligned}$$

Denote by $L$ the straight line containing the ray $l$. We have $l \cap K = \varnothing$, since $P_Y(l) = P_N(l) = z$. By Lemma 1 $z \in Z_N(K)$.

We see that $K$ is convex. Closed and bounded, and $l$ is boundedly compact, hence, applying the Hahn–Banach theorem we get a hyperplane $H_1$ strictly separating $l$ and $K$. Let $z_1 = P_{H_1}(z)$.

We shall prove that $z_1 \in N$. If $L$ and $H_1$ are parallel, then by the orthogonality of $L$ and $H$ we have $z_1 \in H \subset N$. Let $L \cap H_1 \neq \varnothing$. Since $H_1$ strictly separates $l$ and $K$. $L \cap H_1$ is a singleton. If $L \cap H_1 = x$, we have $x \notin l$, $\langle f, x\rangle < \langle f, z\rangle$. Furthermore

$$x \in H_1, \qquad z_1 = P_{H_1}(z);$$

hence

$$\langle x - z_1, z - z_1\rangle = 0,$$

and then

$$\begin{aligned}\|x-z\|^2 &= \langle x-z, x-z\rangle \\ &= \langle z-z_1, z-z_1\rangle + 2\langle x-z_1, z-z_1\rangle + \langle z-z_1, z-z_1\rangle \\ &= \|z-z_1\|^2 + \|x-z_1\|^2;\end{aligned}$$

hence

$$\|x-z\| > \|x-z_1\|.$$

Then

$$\begin{aligned}\langle f, z_1\rangle &= \langle f, z\rangle + \langle f, x-z\rangle + \langle f, z_1-x\rangle \\ &= \langle f, z\rangle - \|x-z\| + \langle f, z-x\rangle \\ &\leqslant \langle f, z\rangle - \|x-z\| + \|z_1 x\| < \langle f, z\rangle, \qquad z_1 \in N.\end{aligned}$$

Let $H_2$ be the hyperplane which is parallel to $H_1$ and passes through an arbitrary point $y \in K$, $P_{H_2}(z) = v$. Then

$$\langle z-v, v-y\rangle = 0, \qquad \|z-v\| = \|z-z_1\| + \|z_1-v\|.$$

We have

$$\begin{aligned}\|z-y\|^2 &= \langle z-y, z-y\rangle \\ &= \langle z-v, z-v\rangle + 2\langle z-v, v-y\rangle + \langle v-y, v-y\rangle \\ &= \|z-v\|^2 + \|v-y\|^2 \\ &= \|z-z_1\|^2 + 2\,\|z-z_1\|\,\|z_1-v\|^2 + \|v-y\|^2 \\ &> \|z_1-v\|^2 + \|v-y\|^2 \\ &= \|z_1-y\|^2,\end{aligned}$$

but $z = Z_N(K)$, a contradiction.

Implication (ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Let $X$ be not strictly convex, $[w_1, w_2] \subset S$, $w_1 \neq w_2$. Take $Y = \operatorname{span}\{w_1, w_2\}$ and $K = [w_2, -w_1]$. We have $\|w_1 + w_2\| = 2$; hence $0 \in Z_Y(K)$; $0 \notin \overline{P_Y(K) = K}$.

If $X$ is strictly convex and non-Hilbert, then by Lemma 2 there exist three points $x_1, x_2, x_3 \in X$ such that if

$$L = \operatorname{span}\{x_1, x_2, x_3\}, \qquad N = \operatorname{conv}\{x_1, x_2, x_3\},$$

then

$$0 \in Z_L(N), \qquad 0 \notin \operatorname{aff}\{x_1, x_2, x_3\}.$$

Let

$$v = (x_1 + x_2 + x_3)/3, \qquad f \in S^*, \qquad f(x_1 - v) = \|f\| \, \|x_1 - v\|,$$
$$Y = \{w \in L : f(w) = 0\}.$$

Since $X$ is strictly convex and $f(x_1 - v) = \|f\| \, \|x_1 - v\|$, we have

$$P_Y(N) \subset \operatorname{aff}\{x_1, x_2, x_3\}.$$

Then

$$0 \in Z_Y(N) \backslash \overline{P_Y(N)}, \qquad \dim Y = 2, \qquad \dim N = 2.$$

## 2. CHARACTERIZATION OF CHEBYSHEV CENTERS IN HILBERT SPACES

Propositions 1 and 2 below are generalizations of Propositions 2.4 and 2.5 in [1], respectively. We apply the sets $P_Y(\overline{\operatorname{conv}}\, A)$ instead of $\overline{\operatorname{conv}}\, (P_Y(A))$ used in [1]. These sets are different in general, however, they coincide if $Y$ is a closed subspace, as in [1].

PROPOSITION 1. *Let $X$ be a Hilbert space, $Y \in \mathscr{F}(X)$ a convex set, $K \in \mathscr{F}(X)$ a convex bounded set, $y \in Y$, and $r = R(y, K)$. Then $y = Z_Y(K)$ if and only if $y \in P_Y(\overline{\operatorname{conv}}(K \backslash B(y, t)))$ for every $0 < t < r$.*

*Proof.* Let $y = Z_Y(K)$. By Proposition 2.3 [1] for every $0 < t < r$, $y = Z_Y(K \backslash B(y, t)) = Z_Y(\overline{\operatorname{conv}}(K \backslash B(y, t)))$. By Theorem 1 $y \in P_Y(\overline{\operatorname{conv}}(K \backslash B(y, t)))$.

Assume now that $y \in P_Y(\overline{\operatorname{conv}}(K \backslash B(y, t)))$ for every $0 < t < r$. Take an arbitrary point $z \in Y \backslash \{y\}$. We shall prove that $R(z, K) > R(y, K)$. By assumption there exists $w(t) \in \overline{\operatorname{conv}}(K \backslash B(y, t))$ such that $P_Y(w(t)) = y$. Then $\langle z - y, w(t) - y \rangle \leqslant 0$. For every number $\varepsilon > 0$ there exists a point $x(t) \in K \backslash B(y, t)$ such that $\langle z - y, x(t) - y \rangle \leqslant \varepsilon$. We have

$$\langle z - x(t), z - x(t) \rangle = \|z - y\|^2 + \|x(t) - y\|^2 - 2\langle z - y, x(t) - y \rangle$$
$$\geqslant \|z - y\|^2 + t^2 - 2\varepsilon.$$

Since $t$ and $\varepsilon$ can be chosen close enough to $r$ and 0, respectively, we have

$$R^2(z, L) \geqslant \|z - y\|^2 + R^2(y, K), \qquad R(z, K) \geqslant R(y, K).$$

As $z \in Y$ is arbitrary, we have $y \in Z_Y(K)$.

PROPOSITION 2. *Let $X$ be a Hilbert space, $Y \in \mathscr{F}(X)$ a convex set, $K \in \mathscr{F}(X)$ a convex compact set, $y \in Y$, and $r = R(y, K)$. Then $y = Z_Y(K)$ if and only if $y \in P_Y(\overline{\text{conv}}(K \cap S(y, r)))$.*

*Proof.* Let $y = Z_Y(K)$. By Proposition 2.2 [1] $y = Z_Y(K \cap S(y, r)) = Z_Y(\overline{\text{conv}}(K \cap S(y, r)))$. By Theorem 1 $y \in P_Y(\overline{\text{conv}}(K \cap S(y, r)))$.

Now let $y \in P_Y(\overline{\text{conv}}(K \cap S(y, r)))$. Then for every $0 < t < r \in P_Y(\overline{\text{conv}}(K \setminus B(y, t)))$. By Proposition 1 $y = Z_Y(K)$.

THEOREM 2. *Let $X$ be a Banach space,* $\dim X \geqslant 3$. *The following statements are equivalent*:

(i) *$X$ is a Hilbert space.*

(ii) *For every bounded convex set $K \in \mathscr{F}(X)$, for every point $x \in Z_K(K)$, and for every number $t$ such that $0 < t < R(x, K)$ we have $x \in \overline{\text{conv}}(K \setminus B(x, t))$.*

(iii) *For every bounded convex set $K \in \mathscr{F}(X)$ and for all $x, t$ such that $0 < t < R(x, K)$, $x \in \overline{\text{conv}}(K \setminus B(x, t))$, we have $x \in Z_K(K)$.*

*Proof.* The implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are true by Proposition 1 (when $Y = K$).

(ii) $\Rightarrow$ (i). If $X$ is not a Hilbert space, then by Lemma 2 there are points $x_1, x_2, x_3 \in X$ such that for $K = \overline{\text{conv}}\{0, x_1, x_2, x_3\}$ we have

$$0 \in Z_K(K), \qquad 0 \notin \text{aff}\{x_1, x_2, x_3\} = L.$$

Clearly, for $N(t) = \overline{\text{conv}}(K \setminus B(0, t))$ we have

$$\sup_{v \in N(t)} \inf_{w \in L} \|v - w\| \to 0$$

as $t \to R(0, K)$, and (ii) fails.

(iii) $\Rightarrow$ (i). If $X$ is not a Hilbert space, then by Lemma 2

$$R(0, K) < \inf\{R(v, K) : v \in N\} = R,$$

where

$$K = \overline{\text{conv}}\{0, x_1, x_2, x_3\}, N = \overline{\text{conv}}\{x_1, x_2, x_3\}.$$

For every point $z \in N$ we have

$$\|z\| \leqslant \max\{\|x_1\|, \|x_2\|, \|x_3\|\} \leqslant R(0, K) < R. \tag{1}$$

Since $N$ is compact and the function $R(\cdot, N)$ is continuous, $Z_N(N)$ is nonempty. Let $w_1 \in Z_N(N)$. Clearly,

$$R(w_1, N) = R, \qquad S(w_1, R) \cap \{x_1, x_2, x_3\} \neq \varnothing.$$

Without loss of generality we may assume that $x_1 \in S(w_1, R)$. If $\|x_2 - w_1\| < R$. $\|x_3 - w_1\| < R$, then for a point $w_2 \in (w_1, x_1)$ such that $\|w_2 - w_1\|$ is small enough we have $\|x_i - w_2\| < R$ $(i = 1, 2, 3)$. Hence, $R(w_2, N) < R$, but $w_1 \in Z_N(N)$, $w_2 \in N$, a contradiction. Without loss of generality we may assume that $x_2 \in S(w_1, R)$. If $x_3 \in S(w_1, R)$ then, by (1) and (iii), $w_1 \in Z_K(K)$, which is impossible. Consider the case $\|x_3 - w_1\| < R$. Let $w_3 = (x_1 + x_2)/2$. By the triangle inequality we have $\|x_1 - x_2\| \leqslant 2R$. If $\|x_1 - x_2\| < 2R$, then $\|x_1 - w_3\| < R$; hence. $\|x_1 - w\| < R$ for an arbitrary point $w \in (w_1, w_3)$; by analogy, $\|x_2 - w\| < R$. If $\|w - w_1\|$ is small enough, we have $\|x_3 - w\| < R$, a contradiction in view of $w_1 \in Z_N(N)$, $w \in N$. Let $\|x_1 - x_2\| = 2R$. If $\|x_3 - w_3\| \leqslant R$, then by inequality (1) for the point $w_3$ condition (iii) is not true. If $\|x_3 - w_3\| > R$, then by the inequality $\|x_3 - w_1\| < R$ there exists a point $w \in (w_1, w_3)$ with $\|x_3 - w\| = R$. We have

$$\|x_1 - w\| \leqslant \max\{\|x_1 - w_3\|, \|x_1 - w_1\|\} = R;$$

similarly, $\|x_2 - w\| \leqslant R$; further

$$2R = \|x_1 - x_2\| \leqslant \|x_1 - w\| + \|x_2 - x\| \leqslant 2R;$$

hence

$$\|x_1 - w\| = \|x_2 - w\| = \|x_3 - w\| = R.$$

As above, (iii) fails for $x = w$.

*Remark.* Theorem 2 is also true, if we, in addition, assume $K$ to be compact and replace the condition "$x \in \overline{\mathrm{conv}}(K \backslash B(x, t))$" by the condition "$x \in \overline{\mathrm{conv}}(K \cap S(x, R(x, M))$." Moreover, one can replace the relation "$x \in Z_K(K)$" by "$x \in Z(K)$."

## REFERENCES

1. D. Amir and J. Mach, Chebyshev centers in normed spaces, *J. Approx. Theory* **40** (1984), 364–374.
2. A. L. Garkavi, The Chebyshev center and the convex hull of a set, *Uspekhi Mat. Nauk* **19** (1964), 139–145.